MATH 17 – PRACTICE EXAM
MIDTERM I

Name: SOLUTIONS

NO CALCULATORS

1. Compute the following integral.

$$\int x \ln x \, dx$$

Integrate by parts with $u = \ln x$, $dv = x\,dx$
$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \boxed{\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C}$$

2. Compute the following integral.

$$\int_0^{\sqrt{3}} x^3 \sqrt{x^2+1}\, dx$$

Substitute $u = x^2+1$. Then $du = 2x\,dx$ and $u-1 = x^2$.

Lower bound: $u = 0^2+1 = 1$
Upper bound: $u = (\sqrt{3})^2+1 = 4$.

$$\frac{1}{2}\int_1^4 (u-1)\sqrt{u}\,du = \frac{1}{2}\int_1^4 u^{3/2} - u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^4$$

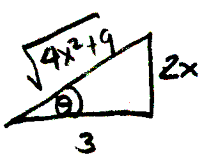
$$= \frac{1}{2}\left(\left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right)$$

$$= \frac{32}{5} - \frac{8}{3} - \frac{1}{5} + \frac{1}{3} = \frac{31}{5} - \frac{7}{3} = \frac{93-35}{15} = \boxed{\frac{58}{15}}$$

3. Compute the following integral.

$$\int \frac{1}{(4x^2+9)^{3/2}}\,dx$$

Let $x = \frac{3}{2}\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\,d\theta$. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$$\frac{3}{2}\int \frac{\sec^2\theta\,d\theta}{(9\sec^2\theta)^{3/2}} = \frac{3}{2}\int \frac{\sec^2\theta}{27\sec^3\theta}\,d\theta$$

$$= \frac{1}{18}\int \cos\theta\,d\theta = \frac{1}{18}\sin\theta + C.$$

$$= \boxed{\frac{x}{9\sqrt{4x^2+9}} + C}$$

4. Compute the following integral.

$$\int \frac{2-4x}{(x^2+1)(x-1)^2}\,dx$$

$$\frac{2-4x}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2 = 2-4x$$

Plug in $x=1$: $2B = -2 \rightarrow B = -1$.

$$A(x-1)(x^2+1) + (Cx+D)(x-1)^2 = 3-4x+x^2$$

Take the derivative:

$$A(x^2+1) + A(x-1)\,2x + (Cx+D)\cdot 2(x-1) + (x-1)^2\cdot C = -4+2x.$$

Plug in 1: $2A = -\cancel{2}_{2} \rightarrow A = -1$.

So $(Cx+D)(x-1)^2 = 3-4x+\cancel{x^2}+x^3-x^2+x-1$

$$Cx^3 + (D-2C)x^2 + (2D-C)x + D = x^3 - 3x + 2$$

$$C=1,\ D=2$$

4 continued.

So we have

$$\int \frac{2-4x}{(x^2+1)(x-1)^2}\,dx = -\int \frac{1}{x-1}\,dx + -\int \frac{1}{(x-1)^2}\,dx + \int \frac{x+2}{x^2+1}\,dx$$

$$= -\ln|x-1| + \frac{1}{x-1} + \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$= \boxed{-\ln|x-1| + \frac{1}{x-1} + \frac{\ln|x^2+1|}{2} + 2\tan^{-1}x + C}$$

5. Determine whether the following integral converges or diverges. If it converges, find its value.

$$\int_0^{\pi} \frac{1}{\cos^2\theta}\,d\theta$$

This function $\left(\frac{1}{\cos^2\theta}\right)$ is discontinuous at $\frac{\pi}{2}$.

$$\int_0^{\pi} \frac{1}{\cos^2\theta}\,d\theta = \int_0^{\pi/2} \sec^2\theta\,d\theta + \int_{\pi/2}^{\pi} \sec^2\theta\,d\theta$$

$$= \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec^2\theta\,d\theta + \lim_{s\to\frac{\pi}{2}^+} \int_s^{\pi} \sec^2\theta\,d\theta$$

$$= \lim_{t\to\frac{\pi}{2}^-} \tan\theta\Big|_0^t + \lim_{s\to\frac{\pi}{2}^+} \tan\theta\Big|_s^{\pi} = \lim_{t\to\frac{\pi}{2}^-}(\tan t - \tan 0) + \lim_{s\to\frac{\pi}{2}^+}(\tan\pi - \tan s)$$

$$= \infty + \infty \quad \boxed{\text{Diverges}}$$

6. Use the trapezoid rule and 4 subintervals to estimate the following integral. Then use Simpson's rule and 4 subintervals to estimate the same integral.

$$\int_{-\pi/2}^{\pi/2} \cos t\,dt$$

$-\frac{\pi}{2}$ $\quad -\frac{\pi}{4}$ $\quad 0$ $\quad \frac{\pi}{4}$ $\quad \frac{\pi}{2}$

Trapezoid rule:

$$T = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

$$= \frac{\pi}{8}\left(\cos\left(-\tfrac{\pi}{2}\right) + 2\cos\left(-\tfrac{\pi}{4}\right) + 2\cos(0) + 2\cos\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{8}\left(0 + \sqrt{2} + 2 + \sqrt{2} + 0\right) = \boxed{\frac{\pi(1+\sqrt{2})}{4}}$$

Simpson's rule:

$$S = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

$$= \frac{\pi}{12}\left(0 + 2\sqrt{2} + 2 + 2\sqrt{2} + 0\right)$$

$$= \boxed{\frac{\pi(1+2\sqrt{2})}{6}}$$